Selected Problems from *Calculus*

by Jon Rogawski

- The problems in this booklet follow the table of contents for the late transcendentals version of the textbook. The section references may be different in the early transcendentals version.
- This booklet is printed in black and white, but the final textbook will feature full-color design. The art will be refined and redrawn in full color as well.
- For a printable copy of these problems, visit the following URL: **http://www.whfreeman.com/rogawskipreview**. Click on "Sample Problems."

Chapter 1 ✧ Pre-Calculus Review

✦ Preliminary Questions

from Section 1.4, The Basic Classes of Functions

1. What is unusual about the domain of the composite $f \circ g$ for $f(x) = x^{\frac{1}{2}}$ and $g(x) = -1 - |x|$?

✦ Exercises

from Section 1.2, Functions, Equations, and Graphs

2. [1] Define $f(x)$ to be the larger of x and $2-x$. Sketch the graph of $f(x)$. What are its domain and range? Express $f(x)$ in terms of the absolute value function.

[1] Adapted from *Calculus Problems for a New Century*, MAA 1993, p. 9

from Section 1.3, Linear and Quadratic Functions

3. Materials expand when heated. If a metal rod has length L_0 at temperature T_0 and if the temperature changes by an amount ΔT, then the length of the rod changes by $\Delta L = \alpha L_0 \Delta T$ where α is the **thermal expansion coefficient** (in units of (degrees)$^{-1}$). Assume the rod is made of steel, for which $\alpha = (1.24)10^{-5}$ per degree Celsius.

(a) If the rod has length $L_0 = 40$ cm at $T_0 = 40°$ C, what is its length at $T = 90°$ C?
(b) If the rod has length $L_0 = 40$ cm at $T_0 = 100°$ C, what is its length at $T = 50°$ C?
(c) Express the rod's length L as a function of T if $L_0 = 40$ cm at $T_0 = 40°$ C.

from Section 1.4, The Basic Classes of Functions

4. The population (in millions) of a country as a function of time t (years) is $P(t) = 30 \cdot 2^{kt}$ with $k = .1$. Show that the population doubles every 10 years. More generally, show that for any non-zero constants a and k, the function $g(t) = a2^{kt}$ doubles after $1/k$ years.

✦ Further Insights and Challenges

from Section 1.5, Trigonometric Functions

5. Apply the double-angle formula to prove:

(a) $\cos \frac{\pi}{8} = \frac{1}{2}\sqrt{2+\sqrt{2}}$

(b) $\cos \frac{\pi}{16} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$

(c) Guess the values of $\cos \frac{\pi}{32}$ and of $\cos \frac{\pi}{2^n}$ for all n.

Chapter 2 ✧ Limits

✦ Preliminary Questions

from Section 2.2, Limits: A Numerical and Graphical Approach

1. Can $f(x)$ approach a limit as $x \to c$ even if $f(c)$ is undefined? If so, give an example.

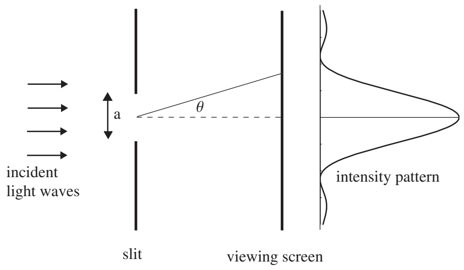
✦ Exercises

from Section 2.2, Limits: A Numerical and Graphical Approach

2. Light waves of frequency λ passing through a slit of width a produce a **Fraunhofer diffraction** pattern of light and dark fringes (see the figure below). The intensity as a function of the angle θ is given by

$$I(\theta) = I_m \cdot \left(\frac{\sin(R\sin\theta)}{R\sin\theta}\right)^2$$

where I_m is a constant and $R = \pi a/\lambda$. Show that the intensity function is not defined at $\theta = 0$. Then check numerically that $I(\theta)$ approaches I_m as $\theta \to 0$ for two values of R (for example, choose any two integer values).



Fraunhofer diffraction pattern

from Section 2.3, Basic Limit Laws

3. Give an example where $\lim_{x\to 0} (f(x)+g(x))$ exists but neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

from Section 2.4, Limits and Continuity

4. Sawtooth Function Draw the graph of $f(x) = x - [x]$. At which points is f discontinuous? Is it left- or right-continuous at those points?

✦ Further Insights and Challenges

from Section 2.6, Trigonometric Limits

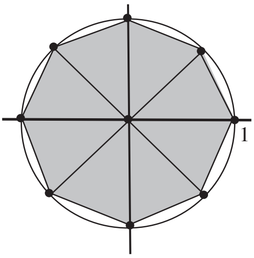
(This problem emphasizes expressing calculus in words.)

5. R & W Let $A(n)$ be the area of a regular n-gon inscribed in a unit circle.

(a) Prove that $A(n) = \frac{1}{2} n \sin\left(\frac{2\pi}{n}\right)$

(b) Intuitively, why might we expect $A(n)$ to converge to the area of the unit circle as $n \to \infty$.

(c) According to Theorem 2, $\lim_{h\to 0} \frac{\sin h}{h} = 1$. Use this to evaluate $\lim_{n\to\infty} A(n)$.



Regular n-gon inscribed in a unit circle.

Chapter 3 ✧ Differentiation

✦ Preliminary Questions

from Section 3.5, Higher Derivatives

1. On September 4, 2003, the Wall Street Journal printed the headline "*Stocks Go Higher, Though the Pace of Their Gains Slows*". Rephrase it as a statement about the first and second time derivatives of stock prices and sketch a possible graph.

✦ Exercises

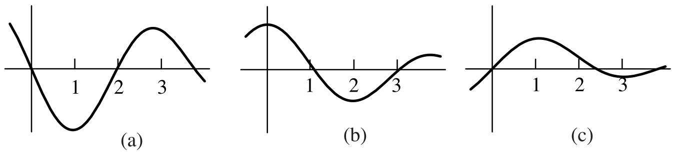
from Section 3.1, Definition of the Derivative

2. [2] Suppose that $y = 5x + 2$ is the equation of the tangent line to the graph of $y = f(x)$ at $a = 3$. What is $f(3)$? What is $f'(3)$?

[2] Suggested by Dennis DeTurck, University of Pennsylvania

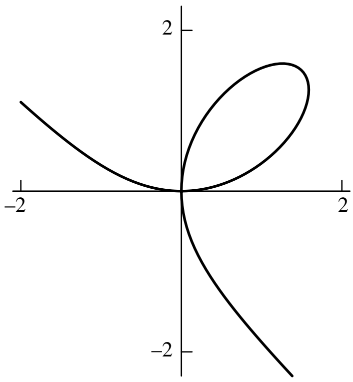
from Section 3.5, Higher Derivatives

3. This figure shows f, f' and f''. Determine which is which.



from Section 3.8, Implicit Differentiation

4. The *folium of Descartes* is the curve with equation $x^3 + y^3 = 3xy$. It was first discussed in 1638 by the French philosopher-mathematician René Descartes, who chose the name *folium*, which means leaf. Descartes' scientific colleague Gilles de Roberval called it the *jasmine flower*. Both men believed incorrectly that the leaf shape in the first quadrant was repeated in each quadrant, giving the appearance of four petals of a flower. Find the equation of the tangent line to this graph at the point $(\frac{2}{3}, \frac{4}{3})$.

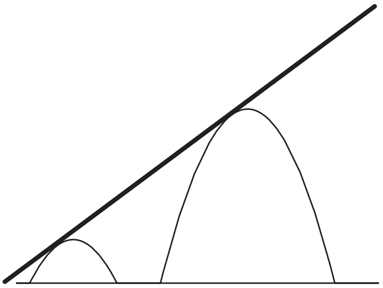


Folium of Descartes: $x^3 + y^3 = 3xy$.

✦ Further Insights and Challenges

from Section 3.2, The Derivative as a Function

5. [3] Two small arches have the shape of parabolas. The first is given by $f(x) = 1 - x^2$ for $-1 \leq x \leq 1$ and the second by $g(x) = 4 - (4 - x)^2$ for $2 \leq x \leq 6$. A board is placed on top of these arches as in the figure below. What is the slope of the board? Hint: find the tangent line to $y = f(x)$ that intersects $y = g(x)$ in exactly one point.



[3] Suggested by Chris Bishop, SUNY Stony Brook.

Chapter 4 ✧ Applications of the Derivative

✦ Preliminary Questions

from section 4.1, Linear Approximation and Applications

1. Discuss how the Linear Approximation makes the following statement more precise: *the sensitivity of the output to a small change in the input depends on the derivative*.

✦ Exercises

from Section 4.6, Applied Optimization

2. Problem of Tartaglia (1500–1557) Among all positive numbers a, b whose sum is 8, find those for which the product of the two numbers and their difference is largest. *Hint:* The quantity to be maximized is abx where $x = a - b$. Write this quantity in terms of x alone and find its maximum.

3. Kepler's Wine Barrel Problem The following problem was stated and solved in a work entitled *Nova stereometria doliorum vinariorum* (Solid Geometry of a Wine Barrel), published in 1615 by the astronomer Johannes Kepler (1571–1630). What are the dimensions of the cylinder of largest volume that can be inscribed in the sphere of radius R?

from Section 4.8, Antiderivatives

(This problem requires a graphing utility.)

4. GU

(a) Use a graphing device to graph $f(x) = 2\sqrt{1.5 - \cos x} - \sqrt{1 + x^2}$ on the interval $[0, 2\pi]$.

(b) Use the properties of the graph to sketch the antiderivative $F(x)$ of $f(x)$ on $[0, 2\pi]$ satisfying $F(0) = 1$. Notice that we cannot find an explicit formula for $F(x)$ because $\sqrt{1.5 - \cos x}$ does not have an elementary antiderivative.

✦ Further Insights and Challenges

from Section 4.8, Antiderivatives

5. Suppose that $F'(x) = f(x)$.

(a) Show that $\frac{1}{2}F(2x)$ is an antiderivative of $f(2x)$.

(b) Find the general antiderivative of $f(kx)$ for any constant k.

Chapter 5 ✧ The Integral

✦ Preliminary Questions

from Section 5.2, The Definite Integral

1. Explain graphically why $\int_0^{\pi} \cos x\, dx = 0$.

✦ Exercises

from Section 5.2, The Definite Integral

(This problem requires a graphing utility.)

2. GU [4] Prove that $\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx \le \frac{\sqrt{2}}{2}$. Hint: graph $(\sin x)/x$ and observe that it is decreasing on $[\pi/4, \pi/2]$.

[4] Suggested by Dennis DeTurck, University of Pennsylvania

from Section 5.3, The Fundamental Theorem of Calculus, Part I

3. Does $\int_0^1 x^n\,dx$ get larger or smaller as n increases? Explain graphically.

from Section 5.4, The Fundamental Theorem of Calculus, Part II

4. [5] Determine $f(x)$ assuming that $\int_0^x f(t)\,dt$ is equal to $x^2 + x$.

[5] Adapted from *Calculus Problems for a New Century*, p. 102.

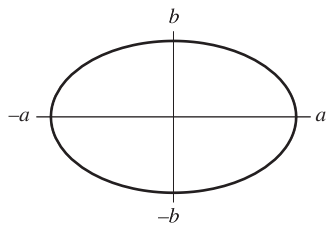
✦ Further Insights and Challenges

from Section 5.6, Substitution Method

5. Area of an Ellipse Prove the formula $A = \pi ab$ for the area of the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Hint: Show that $A = 2b \int_{-a}^{a} \sqrt{1 - (x/a)^2}\,dx$, change variables, and use the formula for the area of a circle.



Graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Chapter 6 ✧ Applications of the Integral

✦ Preliminary Questions

from Section 6.4, The Method of Cylindrical Shells

1. Let V be the volume of a solid of revolution about the y-axis.

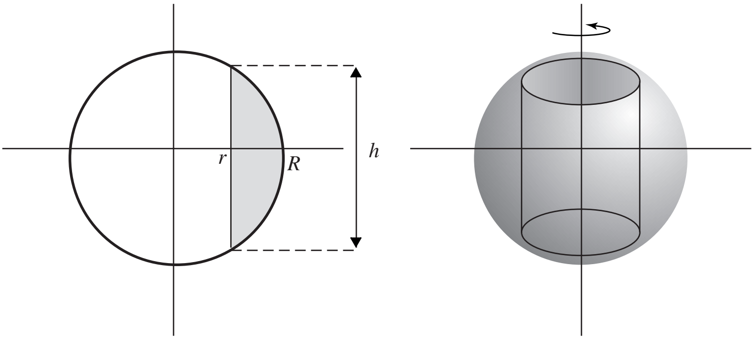
(a) Does the Shell Method for computing V lead to an integral with respect to x or y?

(b) Does the Disk or Washer Method for computing V lead to an integral with respect to x or y?

✦ Exercises

from Section 6.3, Volumes of Revolution

2. A *bead* is formed by removing a cylinder of radius r from the center of a sphere of radius R (see figure below). Find the volume of the *bead* with $r = 1$ and $R = 2$.



A bead is a sphere with a cylinder removed.

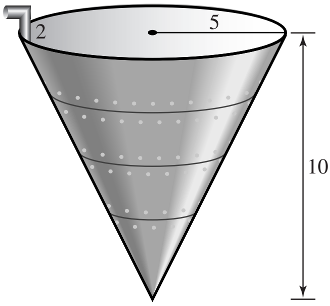
from Section 6.4, The Method of Cylindrical Shells

3. Use both the Shell and the Disk Methods to calculate the volume of the solid obtained by rotating the region under the graph of $f(x) = 8 - x^3$ for $0 \le x \le 2$ about:

(a) the x-axis **(b)** the y-axis

from Section 6.5, Work and Energy

4. Calculate the work (in joules) required to pump water out through the spout of the conical tank in the figure below. Assume the tank is full, distances are in meters, and the density of water is 1000 kg/m^3.



✦ Further Insights and Challenges

from Section 6.3, Volumes of Revolution

(This problem emphasizes expressing calculus in words. Refer to the figure on page 7 of this booklet.)

5. R & W Find the volume V of the bead (see the first figure on page 7 of this booklet) in terms of r and R. Then show that $V = \frac{\pi}{6} h^3$ where h is the height of the bead. This formula has a surprising consequence: since V can be expressed in terms of h alone, it follows that two beads of height 2 inches, one formed from a sphere the size of an orange and the other the size of the earth would have the same volume![6] Can you explain intuitively how this is possible?

[6] G. Alexanderson and L. Klosinski, "Some Surprising Volumes of Revolution," *Two-Year College Mathematics Journal*, v. 6, No. 3, 1975, pp. 13–15.

Chapter 7 ✧ The Exponential Function

✦ Preliminary Questions

from Section 7.2, Inverse Functions

1. A homework problem asks for a sketch of the graph of the *inverse* of $f(x) = x + \cos x$. Frank, after trying but failing to find a formula for $f^{-1}(x)$, says it's impossible to graph the inverse. Sally hands in an accurate sketch without solving for f^{-1}. How did Sally do it?

✦ Exercises

from Section 7.5, Compound Interest and Present Value

(This problem emphasizes expressing calculus in words.)

2. R & W **Banker's Rule of 70** Bankers have a rule of thumb: if you receive R percent interest, continuously compounded, then your money doubles after approximately $70/R$ years. For example, at $R = 5\%$, your money doubles after $70/5$ or 14 years. Use the concept of doubling time to justify the Banker's Rule. (Note: sometimes the approximation $72/R$ is used. It is less accurate but easier to apply because 72 is divisible by more numbers than 70).

from Section 7.6, Models Involving $y' = k(y - b)$

3. A hot metal bar is submerged in a large reservoir of water whose temperature is 60° F. The temperature of the bar 20 seconds after submersion is 100° F. After one minute, the temperature has cooled to 80° F.

(a) Determine the cooling constant k.

(b) What is the differential equation satisfied by the temperature $F(t)$ of the bar?

(c) What is the formula for $F(t)$?

(d) Determine the temperature of the bar at the moment it is submerged.

from Section 7.7, L'Hôpital's Rule

4. Assumptions Matter[7] Let $f(x) = x(2 + \sin x)$ and $g(x) = x^2 + 1$.

(a) Show directly that $\lim_{x\to\infty} f(x)/g(x) = 0$

(b) Show that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ but $\lim_{x\to\infty} f'(x)/g'(x)$ does not exist.

Do (a) and (b) contradict L'Hôpital's Rule? Explain.

[7] Adapted from *Some subtleties in L'Hôpital's Rule* by Robert J. Bumcrot, reprinted in *A Century of Calculus, Part II*, MAA 1992

✦ Further Insights and Challenges

from Section 7.3, Logarithms and Their Derivatives

5. Defining $\ln x$ **as an Integral.** Define a function $\varphi(x)$ in the domain $x > 0$:

$$\varphi(x) = \int_1^x \frac{1}{t}\, dt$$

This exercise proceeds as if we didn't know that $\varphi(x) = \ln x$ and deduces the basic properties of $\ln x$ from the integral expression. Prove the following statements:

(a) $\int_1^b \frac{1}{t}\, dt = \int_a^{ab} \frac{1}{t}\, dt$ for all $a, b > 0$.
Hint: use the substitution $u = t/a$.

(b) $\varphi(ab) = \varphi(a) + \varphi(b)$. *Hint:* break up the integral from 1 to ab into two integrals and use (a).

(c) $\varphi(1) = 0$ and $\varphi(a^{-1}) = -\varphi(a)$ for $a > 0$.

(d) $\varphi(a^n) = n\varphi(a)$ for all $a > 0$ and integers n.

(e) $\varphi(a^{1/n}) = \frac{1}{n}\varphi(a)$ for all $a > 0$ and integers n.

(f) $\varphi(a^r) = r\varphi(a)$ for all $a > 0$ and rational number r.

(g) There exists x such that $\varphi(x) > 1$. *Hint:* show that $\varphi(a) > 0$ for any $a > 1$. Then take $x = a^m$ for $m > 1/\varphi(a)$.

(h) Show that $\varphi(t)$ is increasing and use the Intermediate Value Theorem to show that there exists a unique number e such that $\varphi(e) = 1$.

(i) $\varphi(e^r) = r$ for any rational number r.

Chapter 8 ✧ Techniques of Integration

✦ Preliminary Questions

from Section 8.2, Integration by Parts

1. For each of the following integrals, state whether substitution or Integration by Parts should be used:

$$\int x\cos(x^2)\, dx, \quad \int x\cos x\, dx,$$
$$\int x^2 e^x\, dx, \quad \int x e^{x^2}\, dx$$

✦ Exercises

from Section 8.1, Numerical Integration

2. Scientists estimate arrival times of tsunamis (seismic ocean waves) based on the point of origin P and ocean depths. The speed s of a tsunami in mph is approximately $s = \sqrt{15d}$ where d is the ocean depth in feet.

(a) Let $f(x)$ be the ocean depth x miles from P (in the direction of the coast). Argue using Riemann sums that the time T required for the tsunami to travel M miles towards the coast is:

$$T = \int_0^M \frac{dx}{\sqrt{15f(x)}}$$

(b) Use Simpson's Rule to estimate T if $M = 1000$ and the ocean depths (in ft), measured at 100 mile intervals starting from P are:

13000, 11500, 10500, 9000, 8500, 7000,
6000, 4400, 3800, 3200, 2000

from Section 8.2, Integration by Parts

3. [8] Find $f(x)$ assuming that

$$\int f(x)\,e^x\,dx = f(x)\,e^x - \int x^{-1}e^x\,dx$$

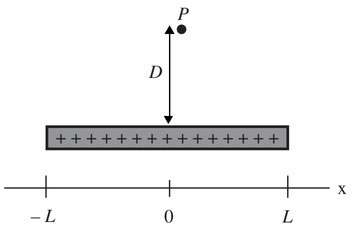
[8] Adapted from *Calculus Problems for a New Century*

from Section 8.4, Trigonometric Substitution

4. A charged wire of length $2L$ creates an electric field at a point P located at a distance D from the midpoint of the wire as in the figure. The strength of the field (in the direction perpendicular to the wire) is equal to

$$E = \int_{-L}^{L} \frac{k\lambda D}{(x^2 + D^2)^{3/2}}\,dx$$

where $k = 8.99 \times 10^9$ N·m^2/C^2 (Coulomb constant) and λ is the charge density (charge per unit length). Find E if $L = 30$ m, $D = 3$ m, and $\lambda = 6 \times 10^{-4}$ C/m.



✦ Further Insights and Challenges

from Section 8.6, Improper Integrals

5. Let p be an integer. Show that $\displaystyle\int_0^{1/2} \frac{dx}{x(\ln x)^p}$ converges if and only if $p > 1$.

Chapter 9 ✧ Further Applications of Integration

✦ Preliminary Questions

from Section 9.1, Arc Length

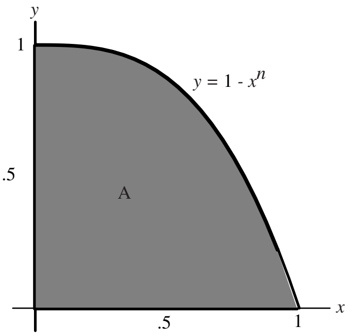
1. How do the arc lengths of the curves $y = f(x)$ and $y = f(x) + C$ differ (C is a constant)? Explain geometrically and then justify using the arc length formula.

✦ Exercises

from Section 9.1, Arc Length

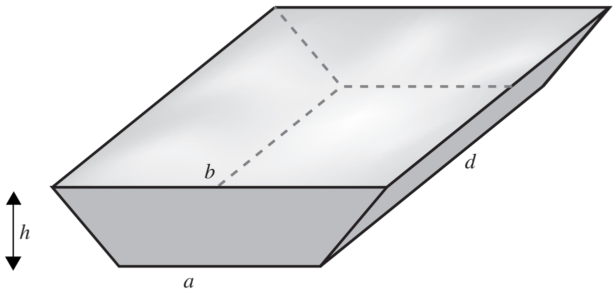
(This problem requires a computer algebra system.)

2. CAS A merchant intends to produce specialty carpets in the shape of the region in the figure, bounded by the graph of $y = 1 - x^n$ for $0 \le x \le 1$ (units in yards). Assume that material costs 50 dollars per square yard and that it costs $50L$ dollars to cut the carpet, where L is the length of the curved side of the carpet. The carpet can be sold for $150A$ dollars, where A is the carpet's area. Find, via numerical integration with a CAS, the whole number n for which the merchant's profits are maximal.



from Section 9.2, Fluid Pressure and Force

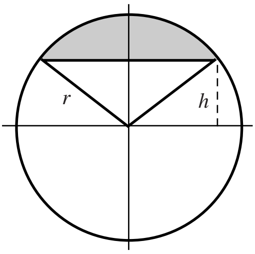
3. The trough in the figure below is filled with corn syrup, whose density is 90 lb/ft^3. Calculate the fluid force on the front side of the trough.



Trough filled with corn syrup (density 90 lb/ft^3).

from Section 9.3, Center of Mass

4. Find the centroid of the shaded subset of the semicircle of radius r in the figure below. What are the coordinates of the centroid when $r = 1$ and $h = \frac{1}{2}$? *Hint:* Use geometry rather than integration to show that the *area* of the region is $r^2 \sin^{-1}(\sqrt{1 - h^2/r^2}) - h\sqrt{r^2 - h^2}$.



✦ Further Insights and Challenges

from Section 9.4, Taylor Polynomials

5. The following equation arises in the description of Bose-Einstein condensation (the quantum theory of gases cooled to near absolute zero):

$$A_0 = \frac{4A}{\sqrt{\pi}} \int_0^\infty \frac{x^2 e^{-x^2}\, dx}{1 - Ae^{-x^2}}$$

and it is necessary to derive an approximate expression for A_0 in terms of A.

(a) Show that the second Maclaurin polynomial for the function $f(A) = \dfrac{x^2 e^{-x^2}}{1 - Ae^{-x^2}}$ (where A is the variable and x is treated as a constant) is:

$$T_2(A) = x^2 e^{-x^2} + Ax^2 e^{-2x^2} + A^2 x^2 e^{-3x^2}$$

(b) Use the approximation $A_0 \approx \dfrac{4A}{\sqrt{\pi}} \displaystyle\int_0^\infty T_2(A)\, dx$ to show:

$$A_0 \approx A + \frac{1}{2\sqrt{2}} A^2 + \frac{1}{3\sqrt{3}} A^3$$

You may use the formula (valid for $\lambda > 0$):

$$\int_0^\infty x^2 e^{-\lambda x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\lambda^3}}$$

Chapter 10 ✧ Introduction to Differential Equations

✦ Preliminary Questions

from Section 10.2, Graphical and Numerical Methods

1. True or False: in the slope field for $\dot{y} = \ln y$, the slopes at points on a vertical line $t = C$ are all equal.

✦ Exercises

from Section 10.1, Solving Differential Equations

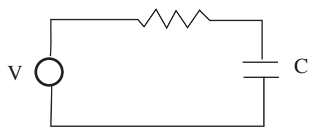
2. The figure below shows a circuit consisting of a resistor of R ohms, a capacitor of C farads, and a battery of voltage V. When the circuit is completed, the amount of charge $q(t)$ (in coulombs) on the plates of the capacitor varies according to the differential equation (t in seconds):

$$R\frac{dq}{dt} + \frac{1}{C}q = V$$

(a) Solve for $q(t)$.

(b) Show that $\lim_{t\to\infty} q(t) = CV$.

(c) Find $q(t)$ assuming that $q(0) = 0$. Show that the capacitor charges to approximately 63% of its final value CV after a period of length $\tau = RC$ (τ is called the *time constant* of the capacitor).



An RC circuit.

from Section 10.3, Population and Other Models

(This problem requires a computer algebra system.)

3. CAS Rainbow trout are harvested from Sunset Lake at a rate of h (in thousands) fish per year. Let $P(t)$ be the trout population (in thousands). To account for harvesting, the logistic equation must be modified as follows:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right) - h$$

Assume that $k = .8$, $A = 20$ and $h = 3$.

(a) Solve for the equilibrium populations and determine if they are stable or unstable.

(b) Determine $\lim_{t\to\infty} P(t)$ for the initial values $P(0) = 4, 7$ and 16.

(c) Use a CAS to draw the slope field.

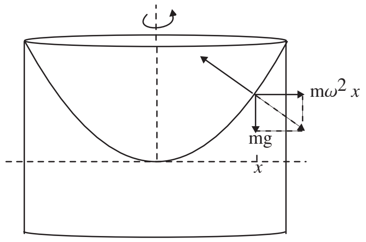
from Section 10.4, First-Order Linear Equations

4. A stream feeds into a lake at a rate of 1000 m^3 per day. Assume the stream is polluted with a toxin whose concentration is 5 grams/m^3. Assume further that the lake has a volume 10^6 m^3 and that water flows out of the lake at the same rate of 1000 m^3 per day. Set up a differential equation for the amount of toxin $M(t)$ (in grams) in the lake at time t and solve for $M(t)$ assuming that $M(0) = 0$. What is the limiting concentration as t gets large?

✦ Further Insights and Challenges

from Section 10.1, Solving Differential Equations

5. [9] If a bucket of water spins about a vertical axis with constant angular velocity ω (in radians per second), the water climbs up the side of the bucket until it reaches an equilibrium position. Two forces act on a water particle located at distance x from the vertical axis: the gravitational force $-mg$ acting in the downward direction and the centrifugal force $m\omega^2 x$ acting in the horizontal direction as indicated in the figure below. The resultant force acts along the direction of the diagonal of the rectangle in the figure and this direction must be normal to the water's surface (that is, perpendicular to the tangent line). Prove that if $y = f(x)$ is the equation of the curve obtained by taking a vertical cross section through the axis, then $-1/y' = -g/(\omega^2 x)$. Show that $y = f(x)$ is a parabola.



[9]Adapted from *Ordinary Differential Equations* by M. Tenenbaum and H. Pollard, Dover, 1985

Notes