# More

## When Is a Physical Quantity Conserved?

The six conservation laws that we have encountered thus far—energy, charge, angular momentum, linear momentum, baryon number, and lepton number—apply to all four of the fundamental interactions. What is required quantum-mechanically for a physical quantity to be conserved?

The time-dependent Schrödinger equation, written in the same form as Equation 6-52, is

$$H_{\text{op}}\Psi = i\hbar \frac{\partial \Psi}{\partial t} \qquad \textbf{13-13}$$

where $\Psi$ is the wave function of the system and $H_{\text{op}}$ is the Hamiltonian (i.e., energy) operator, assumed here to be independent of time. We saw in Section 6-4 that the expectation value of an observable physical quantity $f$ is given by

$$\langle f \rangle = \int_{-\infty}^{+\infty} \Psi^* f_{\text{op}} \Psi \, dx \qquad \textbf{13-14}$$

where $f_{\text{op}}$ is the operator representing the quantity $f$. For $f$ to be conserved, its value $\langle f \rangle$ must not change in time, so the question is, When is $\langle f \rangle$ independent of time? To answer that question we assume that $f_{\text{op}}$ is independent of time and compute $d\langle f \rangle / dt$ as follows:

$$\frac{d\langle f \rangle}{dt} = \frac{d}{dt} \int \Psi^* f_{\text{op}} \Psi \, dx = \int \frac{d\Psi^*}{dt} f_{\text{op}} \Psi \, dx + \int \Psi^* f_{\text{op}} \frac{d\Psi}{dt} \, dx \qquad \textbf{13-15}$$

The complex conjugate Schrödinger equation is

$$(H_{\text{op}}\Psi)^* = -i\hbar \frac{\partial \Psi^*}{\partial t} \qquad \textbf{13-16}\boldsymbol{a}$$

Using the fact that $H_{\text{op}}$ is real,[11] this can be written

$$\Psi^* H_{\text{op}} = (H_{\text{op}}\Psi)^* = -i\hbar \frac{\partial \Psi^*}{\partial t} \qquad \textbf{13-16}\boldsymbol{b}$$

Combining Equations 13-13 and 13-16*b* with 13-15 results in

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{i}{\hbar} \int \Psi^* (H_{\text{op}} f_{\text{op}} - f_{\text{op}} H_{\text{op}}) \Psi \, dx \qquad \textbf{13-17}$$

The quantity in parentheses is called the *commutator* of $H_{\text{op}}$ and $f_{\text{op}}$. We see from Equation 13-17 that $\partial\langle f \rangle / \partial t$ will be zero, that is, $f$ will be conserved, if $[H_{\text{op}} f_{\text{op}} - f_{\text{op}} H_{\text{op}}] = 0$. This occurs if $H_{\text{op}} f_{\text{op}} = f_{\text{op}} H_{\text{op}}$, in which case we say that $H_{\text{op}}$ and $f_{\text{op}}$ commute. Thus, we can state the following rule:

*Continued*

Operators that commute with the Hamiltonian represent conserved physical quantities.

As an obvious example, the Hamiltonian (total energy) operator certainly commutes with itself; therefore, the total energy is a conserved quantity.

Finding such operators is the hard part, since the complete form of $H_{\text{op}}$ is not often known in nuclear and particle physics. As it turns out, conserved quantities can still be found if it can be shown that $H_{\text{op}}$ is invariant under a *symmetry operation* that is related to the physical quantity. As examples, invariance of $H_{\text{op}}$ under translation in space leads to conservation of linear momentum, and invariance of $H_{\text{op}}$ under translation in time leads to conservation of total energy. An acceptable symmetry operator $U_{\text{op}}$ is one which transforms $\Psi$ into $\Psi'$ according to

$$U_{\text{op}}\Psi(x, t) = \Psi'(x, t) \tag{13-18}$$

in such a way that the wave function remains normalized and the new $\Psi'$ satisfies the Schrödinger equation. Determining the form of the connection between $U_{\text{op}}$ and $f_{\text{op}}$ in which $f_{\text{op}}$ corresponds to an observable physical quantity is beyond the level of our discussion, but the result, given initially by H. Weyl, is

$$U_{\text{op}} = e^{ibf_{\text{op}}} \tag{13-19}$$

where $b$ is an arbitrary real quantity independent of $x$ and $t$. A transformation of this form is called a *global gauge transformation*, where "global" means "everywhere" and "gauge" means "scale." Thus, such a transformation changes the measuring scale everywhere at once. If $\Psi'$ also satisfies the Schrödinger equation, as we stated above, then the Schrödinger equation is *gauge-invariant* under the particular symmetry transformation $U_{\text{op}}$; that is, $U_{\text{op}}$ has no effect other than changing the scale everywhere. As a consequence, $f_{\text{op}}$ represents a conserved quantity. The following example is an illustration of how this works.

### Example 13-7 Conservation of Charge

Use a global gauge transformation to show that electric charge is conserved.

#### Solution

$\Psi(x, t)$ describes a system with charge $q$ that satisfies Equation 13-13. If we define the charge operator $Q_{\text{op}}$, the $\langle Q\rangle$ will be conserved if $H_{\text{op}}$ and $Q_{\text{op}}$ commute, that is, if $H_{\text{op}}Q_{\text{op}} = Q_{\text{op}}H_{\text{op}}$. Then $Q_{\text{op}}\Psi = q\Psi$ and charge $q$ will be conserved.

To see that global gauge invariance ensures that $H_{\text{op}}$ and $Q_{\text{op}}$ commute, we write

$$\Psi'(x, t) = e^{ibQ_{\text{op}}}\Psi(x, t) \tag{13-20}$$

where $\Psi'$ also satisfies Equation 13-13, which becomes

$$H_{\text{op}}e^{ibQ_{\text{op}}}\Psi = i\hbar\frac{\partial(e^{ibQ_{\text{op}}}\Psi)}{\partial t} \tag{13-21}$$

Multiplying Equation 13-21 by $e^{-ibQ_{\text{op}}}$ and noting that $Q_{\text{op}}$ is independent of time yields

$$e^{-ibQ_{\text{op}}}H_{\text{op}}e^{ibQ_{\text{op}}}\Psi = i\hbar\frac{\partial(e^{-ibQ_{\text{op}}} \times e^{ibQ_{\text{op}}}\Psi)}{\partial t} = i\hbar\frac{\partial\Psi}{\partial t} \tag{13-22}$$

*Continued*

Comparing Equation 13-22 with Equation 13-13, we see that

$$e^{-ibQ_{\mathrm{op}}}H_{\mathrm{op}}e^{ibQ_{\mathrm{op}}} = H_{\mathrm{op}} \qquad \textbf{13-23}$$

Since $b$ is arbitrary, we select it small enough so $bQ_{\mathrm{op}} \ll 1$ and expand the exponentials in Equation 13-23 in powers of the exponents, keeping only the first terms, to obtain

$$(1 - ibQ_{\mathrm{op}})H_{\mathrm{op}}(1 + ibQ_{\mathrm{op}}) = H_{\mathrm{op}} \qquad \textbf{13-24}$$

Multiplying this out and discarding the second-order terms in $bQ_{\mathrm{op}}$, we have

$$H_{\mathrm{op}}Q_{\mathrm{op}} - Q_{\mathrm{op}}H_{\mathrm{op}} = 0$$

Therefore, $\langle Q \rangle = q$ is conserved. Thus, global gauge invariance ensures the conservation of electric charge.

There are also *local gauge transformations*, where the quantity $b$ in Equation 13-19 is a function of position and time. Although these are mathematically beyond the scope of our discussions, the symmetry of the basic interactions under a number of local gauge transformations leads to several additional conserved quantities. These are discussed in the subsection "More Conservation Laws" in the textbook.